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Simulating POVMs on EPR pairs with 5.7 bits of expected communication

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Abstract. We present a classical protocol for simulating correlations obtained by bipartite POVMs on an EPR pair. The protocol uses shared random variables (also known as local hidden variables) augmented by 5.7 bits of expected communication.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.Hk Quantum communication – 03.67.-a Quantum information

Entanglement simulation was first introduced by Maudlin in a 1992 paper published in a philosophical journal [1] and was revived independently by Brassard, Cleve and Tapp in 1999 [2]. The objective was to quantify the nonlocality [3] of EPR pairs [4] in terms of the amount of communication required to simulate the correlations obtained by bipartite measurement of an EPR pair. "The key to understanding violations of Bell's inequality is not operator algebras but information transmission [1]." This approach increases our understanding of the relationships between classical information and quantum information. It also helps us gauge the amount of information hidden in the EPR pair itself or, in some sense, the amount of information that must be space-like transmitted, in a local hidden variable model, in order for nature to account for the Bell inequalities.

In this scenario, Alice and Bob try to output a and b respectively, through a classical protocol, with the same probability distribution, hence correlations, as if they shared an EPR pair and each measured his or her half of the pair according to a given random von Neumann measurement. In [2], a protocol using an infinite amount of random shared variables and eight bits of communication in the worst case was given. Independently, Steiner [5] presented a protocol using an infinite amount of random shared variables and 1.48 bits of *expected* communication for the simulation of von Neumann measurements in the *real plane*. Surprisingly, Maudlin [1] already had a protocol using random shared variables and 1.17 bits of expected communication also to simulate von Neumann measurements in the real plane. This later result was improved by Cerf, Gisin and Massar [6] to 1.19 bits of expected communication, still with an infinite amount of shared variables, for *arbitrary* von Neumann measurements. They were also able to generalize their protocol to simulate a von Neumann measurement on Alice's part of the EPR pair and a POVM on Bob's part with 6.38 bits of expected communication. It is straightforward to generalize their result to bipartite POVMs with the same amount of communication. Although both type of simulations, worst-case and expected, used shared random variables so far, it was shown by Massar et al. [7] that only protocols of the worst-case communication type required shared randomness. In fact, when considering the simulation of quantum entanglement with a protocol that uses a bounded amount of communication, an infinite amount of shared variables is needed. In [7], a protocol to simulate POVMs with 20 bits of expected communication, without *any* shared randomness, was detailed. Subsequent refinements have been made on the result of Brassard, Cleve and Tapp. Csirik [8] proposed a protocol using only six bits of communication and recently Toner and Bacon [9] presented a protocol using only *one* bit of communication. For a survey, one can be referred to [10].

In this paper, we present a protocol, based on [6,9], to *simulate arbitrary POVMs* on both parts of an EPR pair *using 5.7 bits of expected communication* and an infinite amount of random shared variables. First, we describe what a POVM is, what is the probability distribution of the outputs if we actually make POVMs on a $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ state and display some tools that we will need for the protocol. Then a description of the protocol is given followed by an analysis.

A POVM is a family of matrices $\{B_i\}$ such that $\sum A_i^\dagger A_i = \sum B_i = \mathrm{I}$, where B_i is called a *POVM element.* On qubits, the POVM elements can be expressed, without lost of generality, as the linear sum of one-dimensional

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$$\Pr[a=i,b=j] = \frac{|\vec{a}_i||\vec{b}_j|}{4(4\pi)^2}\iint d\vec{v}_1 d\vec{v}_2 \Theta(-\vec{b}_j\cdot((-1)^c\vec{v}_1+(-1)^d\vec{v}_2))$$
$$= \frac{|\vec{a}_i||\vec{b}_j|}{8} - \frac{|\vec{a}_i||\vec{b}_j|}{8(4\pi)^2}\iint d\vec{v}_1 d\vec{v}_2\, \mathrm{sgn}(\vec{a}_i\cdot\vec{v}_1)\mathrm{sgn}(\vec{b}_j\cdot(\vec{v}_1+\mathrm{sgn}(\vec{a}_i\cdot\vec{v}_1)\mathrm{sgn}(\vec{a}_i\cdot\vec{v}_2)\vec{v}_2)) = \frac{|\vec{a}_i||\vec{b}_j| - \vec{a}_i\cdot\vec{b}_j}{8}. \tag{1}$$

projectors [6,11]. From the spectral decomposition, we know that a POVM element can be written in the form $B=\sum_j b_{ij}P_{ij}$, where b_{ij} are real constants and P_{ij} are projectors. We can then construct a new POVM with the elements $b_{ij}P_{ij}$ and say that if the outcome $b_{ij}P_{ij}$ is chosen that it is actually the outcome B_i that is produced. If the spectral decomposition is not unique, we can include a map from B_i to its rightful decomposition. With this construction, we can focus only on POVMs with elements proportional to projectors. Thus one can find vectors on the Bloch sphere $\vec{b}_i$ such that $B_i = (|\vec{b}_i|\mathrm{I}+\vec{b}_i\cdot\vec{\sigma})/2$, where $\vec{\sigma}$ are the Pauli matrices and with the completeness conditions $\sum|\vec{b}_i| = 2$ and $\sum\vec{b}_i = 0$.

Let's assume that Alice and Bob share a $|\Psi^-\rangle$ state. Alice receives the description of a POVM $\{A_i\}$ and Bob receives the description of a POVM $\{B_j\}$. If they each measure their half of the $|\Psi^-\rangle$ state according to their description of the POVM, Alice will produce $a=i$ with probability $\Pr[a=i] = |\vec{a}_i|/2$, Bob will produce $b=j$ with probability $\Pr[b=j]=|\vec{b}_j|/2$ and the joint probability will be $\Pr[a=i,b=j] = (|\vec{a}_i||\vec{b}_j| - \vec{a}_i\cdot\vec{b}_i)/4$. Now let us turn our attention to the presentation of the protocol. The classical protocol (with 5.7 bits of expected communication) to simulate an arbitrary bipartite POVM on a $|\phi^+\rangle$ can be described as follows:

- Alice and Bob share two random unit vectors $\vec{v}_1,\vec{v}_2\in\mathbb{R}^3$;
- Alice and Bob are given a description of their POVM $\{A_i\}$ and $\{B_i\}$ respectively;
- Alice chooses the ith output of her POVM according to the probability distribution $\Pr[a=i] = |\vec{a}_i|/2$;
- Alice sends $c=\Theta(-\vec{a}_i\cdot\vec{v}_1)$ and $d=\Theta(-\vec{a}_i\cdot\vec{v}_2)$, where $\Theta(x) = \begin{cases}1 \text{ if } x\geq 0\\ 0 \text{ if } x<0\end{cases}$;
- Bob chooses the jth output of his POVM according to the probability distribution $\Pr[b=j] = |\vec{b}_j|/2$;
- Bob checks if $-\vec{b}_j\cdot((-1)^c\vec{v}_1+(-1)^d\vec{v}_2)<0$, if so he sends 0 to Alice and they start over with a fresh set of random variables;
- otherwise, Bob sends 1 to Alice and they produce their output i and j respectively.

The analysis of the protocol is quite simple. The probability of Alice obtaining the POVM outcome i is, as stated, $\Pr[a=i]=|\vec{a}_i|/2$. As for Bob's marginal probability distribution, the vector $(-1)^c\vec{v}_1+(-1)^d\vec{v}_2$ can be considered as a vector pointing in a random direction. Therefore, Bob has probability 1/2 of rejecting $\vec{b}_j$. Since each time around the probabilities are independent, Bob's marginal probability is $\Pr[b=j]=|\vec{b}_j|/2$. For the joint probability distribution, the calculation is a bit tricky but is still straightforward:

see equation (1) above.

The 1/2 factor in (1), is removed by renormalization (which is allowed since all instances are independent from one another) [6]. To realize the protocol Alice must send two bits to Bob and Bob one bit to Alice. Since each round is independent of the preceding ones and each has probability of 1/2 of ending the protocol, the protocol takes an average of two rounds, hence $2(2+1)=6$ bits of expected communication. If we are allowed block-coding, the communication can be lowered. Alice still sends c to Bob, but she can send $d'=\Theta(\vec{a}_i\cdot\vec{v}_1)\oplus\Theta(\vec{a}_i\cdot\vec{v}_2)$ from which d can be easily recovered. From [9], we know that d' can be compressed to an average of 0.85 bits [9]. The communication then becomes $2(1+0.85+1)=5.7$ bits.

Although this result is a small improvement and a simplification of the result in [6], much is left to do. We do not know any way to simulate POVMs with worst-case communication or even if it is possible to do so. While von Neumann measurements can be done with worst-case communication, it is not clear that the same can be done for POVMs. Von Neumann measurements have only two possible outcomes while POVMs have an unbounded number of outcomes. Can a protocol with a bounded amount of communication choose correctly among an unbounded number of outcomes? Can we generalize these types of protocols to simulate arbitrary measurements on n EPR pairs? GHZ and other entangled states?

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